Expressing Exceptionally Large Numbers

Introduction

I have always been fascinated by very large numbers. When I was a child, I learnt what a googol is (10^{100}), what a centillion is (10^{600}) and what a googolplex is ($10\^10^{100}$). More recently, I have heard of Skewes' Number (which is approximately $10\^(10\^10^{34})$). The key to creating huge numbers, such as those bigger than these is by using recursion. For example, if one wants to create a power tower of 10's, he does so by using this iterative formula:

$$t(n+1) \;=\; 10 \wedge t(n), \qquad \text{with starting value } t(0) = 1.$$

Of course, there are finite numbers that are so huge that they cannot even be expressed as power towers of numbers by repeated exponentiation, for example,

$$t(10) \;=\; 10 \wedge (10 \wedge (10 \wedge (10 \wedge (10 \wedge (10 \wedge (10 \wedge (10 \wedge (10 \wedge 10)))))))).$$

In order to express these numbers, a new notation needs to be used.

Knuth's Up-arrow Notation

In 1976, Donald Knuth invented a system for dealing with extremely large numbers. This system involves the use of higher operators following the sequence set by the well known three (addition, multiplication and exponentiation), which are symbolised in the form of up-arrows. The sequence of these operators begins as follows:

$$a + b,$$

$$a \times b \;=\; a + a + a + \ldots\ldots + a \qquad \text{(b terms)},$$

$$a \wedge b \;=\; a^b \;=\; a \times a \times a \times \ldots\ldots \times a \qquad \text{(b terms)},$$

$$a \wedge\wedge b \;=\; a \wedge a \wedge a \wedge \ldots\ldots \wedge a \qquad \text{(b terms)},$$

$$a \wedge\wedge\wedge b \;=\; a \wedge\wedge a \wedge\wedge a \wedge\wedge \ldots\ldots \wedge\wedge a \qquad \text{(b terms)},$$

$$a \wedge\wedge\wedge\wedge b \;=\; a \wedge\wedge\wedge a \wedge\wedge\wedge a \wedge\wedge\wedge \ldots\ldots \wedge\wedge\wedge a \qquad \text{(b terms)},$$

and so on. In general,

$$a \wedge\wedge\wedge^{\cdots\cdots}\wedge\, b \quad \text{(with n arrows or ^s)} \quad \text{which can also be written as}$$

$$a \wedge^{(n)} b \;=\; a \wedge^{(n-1)} a \wedge^{(n-1)} a \wedge^{(n-1)} \ldots\ldots \wedge^{(n-1)} a \qquad \text{(b terms)},$$

so, these higher operators are recursive.

When working with powers and multiple arrows, we start from the right and move leftwards as this produces a number far greater than the one produced by working from left to right. This is because these operations are neither commutative nor associative. For example,

$$a \wedge^{(n+1)} 4 \;=\; a \wedge^{(n)} a \wedge^{(n)} a \wedge^{(n)} a \;=\; a \wedge^{(n)} (a \wedge^{(n)} (a \wedge^{(n)} a)).$$

The operation a^^b (power tower of b a's) is known as a hyper-4 operation and is also known as tetration, superpower and superdegree. The operation a^^^b is known as hyper-5 and a^^^^b is known as hyper-6, and so on.

The power tower of 10 10's shown in the Introduction can therefore be simplified to 10^^10 and further simplified to 10^^^2.

Even with a, b small, as one moves to higher and higher operators in the sequence, the numbers grow with astonishing rapidity. For example:

3 + 3 = 6,
3 × 3 = 9,
3 ^ 3 = 27,
3 ^^ 3 = 3 ^ 3 ^ 3 = 3^{27} = 7,625,597,484,987,
3 ^^^ 3 = 3 ^^ 3 ^^ 3 = 3 ^^ (3 ^ 3 ^ 3) = 3 ^^ 3^{27}
= 3 ^ 3 ^ 3 ^ ...... ^ 3 (with 3^{27} terms),
3 ^^^^ 3 = 3 ^^^ 3 ^^^ 3
= 3 ^^^ (3 ^ 3 ^ 3 ^ ...... ^ 3) (with 3^{27} 3's in power tower)
= 3 ^^ 3 ^^ 3 ^^ ...... ^^ 3
(where the no. of terms is a power tower of 3^{27} 3's).

Incredibly large already after just 4 arrows? Then how about 3^^^^3 arrows? Just to calculate G(1), the first number of a long sequence of numbers!

G(1) = $3 \,\hat{}^{(3\hat{}\hat{}\hat{}\hat{}3)}\, 3$ (contains 3^^^^3 arrows).

Now, G(2) follows on from G(1) and contains G(1) arrows,

G(2) = $3 \,\hat{}^{(G(1))}\, 3$, and this sequence continues,
G(3) = $3 \,\hat{}^{(G(2))}\, 3$,
G(4) = $3 \,\hat{}^{(G(3))}\, 3$, and so on.

In general, G(n+1) = $3 \,\hat{}^{(G(n))}\, 3$ (contains G(n) arrows).

G(n) is a function that is also recursive (like the arrow notation itself) where the next no. in the sequence contains the same no. of arrows as the entire answer of the previous no. We keep going until we reach G(63), for G(63) is known as Graham's Number (G), which is listed in the Guinness Book of Records as the largest number used in any mathematical proof.

However, if the pairs of 3's at the two ends of the strings of arrows were replaced by pairs of 2's, all the answers can be no other than 4, since no matter how many arrows we place between a pair of 2's, the answer is always 4. This is because, for any n,

$$2 \,\hat{}^{(n)}\, 2 \;=\; 2 \,\hat{}^{(n-1)}\, 2 \;=\; 2 \,\hat{}^{(n-2)}\, 2 \;=\; \ldots\ldots \;=\; 2 \,\hat{}^{(2)}\, 2 \;=\; 2 \,\hat{}\, 2 \;=\; 4.$$

Conway’s Chained Arrow Notation

John Conway has gone one step further, in which he invented his chained arrow notation. It operates according to the following rules:

When the chain consists of 3 numbers, then

$$a \to b \to c \;=\; a \,{}^\wedge{}^{(c)}\, b \qquad \text{(with c up-arrows).}$$

If the last no. in the chain is a 1, it can be dropped:

$$a \to b \to \ldots\ldots \to x \to 1 \;=\; a \to b \to \ldots\ldots \to x.$$

If the penultimate no. in the chain is a 1, the last 2 nos. can be dropped:

$$a \to b \to \ldots\ldots \to x \to 1 \to z \;=\; a \to b \to \ldots\ldots \to x.$$

If there are just 2 nos. in the chain, the remaining arrow becomes an exponent:

$$a \to b \to 1 \;=\; a \to b \;=\; a \,{}^\wedge\, b.$$

The last no. in the chain can be reduced by 1 by taking the penultimate no. and replacing it with a copy of the entire chain with its penultimate no. reduced by 1:

$$\begin{aligned}&a \to b \to \ldots\ldots \to x \to (y+1) \to (z+1)\\ &\quad = a \to b \to \ldots\ldots \to x \to (a \to b \to \ldots\ldots \to x \to y \to (z+1)) \to z.\end{aligned}$$

For example:

$$\begin{aligned}&a \to b \to \ldots\ldots \to x \to 2 \to (z+1)\\ &\quad = a \to b \to \ldots\ldots \to x \to (a \to b \to \ldots\ldots \to x \to 1 \to (z+1)) \to z\\ &\quad = a \to b \to \ldots\ldots \to x \to (a \to b \to \ldots\ldots \to x) \to z \qquad \text{(1 nested bracket),}\\ &a \to b \to \ldots\ldots \to x \to 3 \to (z+1)\\ &\quad = a \to \ldots\ldots \to x \to (a \to \ldots\ldots \to x \to 2 \to (z+1)) \to z\\ &\quad = a \to \ldots\ldots \to x \to (a \to \ldots\ldots \to x \to (a \to \ldots\ldots \to x \to 1 \to (z+1)) \to z) \to z\\ &\quad = a \to \ldots\ldots \to x \to (a \to \ldots\ldots \to x \to (a \to \ldots\ldots \to x) \to z) \to z\\ &\qquad\qquad \text{(2 nested brackets).}\end{aligned}$$

In general,

$$\begin{aligned}&a \to b \to \ldots\ldots \to x \to (y+1) \to (z+1)\\ &\quad = a \to \ldots\ldots \to x \to (a \to \ldots\ldots \to x \to (\ \ldots\ldots (a \to \ldots\ldots \to x) \to z) \ldots\ldots\ ) \to z\\ &\qquad\qquad \text{(with y nested brackets (n.b.)),}\end{aligned}$$

so, Conway’s chained arrow notation is yet another example involving the use of recursion.

The brackets can only be removed after the chain inside the brackets has been evaluated into a single number.

The number 3^^^^3 can be expressed in the following form,

$$3 \;\hat{}\hat{}\hat{}\hat{}\; 3 \;=\; 3 \to 3 \to 4.$$

G(1) and larger G(n) numbers can be expressed in the following forms:

$$\begin{aligned}
G(1) &= 3 \to 3 \to (3\hat{}\hat{}\hat{}\hat{}3) \\
&= 3 \to 3 \to (3 \to 3 \to 4) && \text{(1 n.b.),} \\
G(2) &= 3 \to 3 \to (3 \to 3 \to (3 \to 3 \to 4)) && \text{(2 n.b.),} \\
G(3) &= 3 \to 3 \to (3 \to 3 \to (3 \to 3 \to (3 \to 3 \to 4))) && \text{(3 n.b.),} \\
G(n) &= 3 \to 3 \to (3 \to 3 \to (\;\ldots\ldots (3 \to 3 \to 4) \ldots\ldots\;)) && \text{(n n.b.),}
\end{aligned}$$

so, Graham's Number, G = G(63) is as G(n) above but with 63 nested brackets.

The number $3 \to 3 \to 64 \to 2$

$$\begin{aligned}
&= 3 \to 3 \to (3 \to 3 \to 63 \to 2) && \text{(1 n.b.)} \\
&= 3 \to 3 \to (3 \to 3 \to (3 \to 3 \to 62 \to 2)) && \text{(2 n.b.)} \\
&= 3 \to 3 \to (3 \to 3 \to (\;\ldots\ldots (3 \to 3 \to 1 \to 2) \ldots\ldots\;)) && \text{(63 n.b.)} \\
&= 3 \to 3 \to (3 \to 3 \to (\;\ldots\ldots (3 \to 3) \ldots\ldots\;)) && \text{(63 n.b.)} \\
&= 3 \to 3 \to (3 \to 3 \to (\;\ldots\ldots (3 \to 3 \to 27) \ldots\ldots\;)) && \text{(62 n.b.),}
\end{aligned}$$

which means that $3 \to 3 \to 65 \to 2$

$$= 3 \to 3 \to (3 \to 3 \to (\;\ldots\ldots (3 \to 3 \to 27) \ldots\ldots\;)) \qquad \text{(63 n.b.),}$$

and since $27 < 3 \to 3 \to 4 < 3 \to 3 \to 27$,

$$3 \to 3 \to 64 \to 2 \;<\; G \;<\; 3 \to 3 \to 65 \to 2,$$

so, this notation enables Graham's Number to be expressed as a chain of just 4 nos. with the 3rd no. between 64 and 65, even though the 4th no. is only 2.

Just to make things easier and more understandable, we define another recursive function (similar to G(n)) that shows how large these numbers can get:

$$\begin{aligned}
f_{a,b}(0) &= 1, \\
f_{a,b}(1) &= a \,\hat{}\, b = a \to b \to 1 \\
&= a \to b \to 1 \to 2 \\
&\qquad \text{(in fact, when the 3rd no. is a 1, the 4th no. can be any no.),} \\
f_{a,b}(2) &= a \,\hat{}^{(a\hat{}b)}\, b = a \to b \to a^b = a \to b \to (a \to b \to 1 \to 2) \\
&= a \to b \to 2 \to 2, \\
f_{a,b}(3) &= a \,\hat{}^{(f_{a,b}(2))}\, b = a \to b \to (a \to b \to 2 \to 2) \\
&= a \to b \to 3 \to 2, \\
f_{a,b}(4) &= a \,\hat{}^{(f_{a,b}(3))}\, b = a \to b \to (a \to b \to 3 \to 2) \\
&= a \to b \to 4 \to 2,
\end{aligned}$$

$$\begin{aligned} f_{a,b}(c+1) &= a \wedge^{(f_{a,b}(c))} b = a \to b \to (a \to b \to c \to 2) \\ &= a \to b \to (c+1) \to 2, \end{aligned}$$

so, $f_{3,3}(64) < G < f_{3,3}(65)$, where G is Graham’s Number.

To tidy the equations up, we drop the a,b subscript of $f_{a,b}$ and take f to mean $f_{a,b}$. Just changing the 4th no. from 2 to 3, we find:

$$\begin{aligned} a \to b \to 2 \to 3 &= a \to b \to (a \to b \to 1 \to 3) \to 2 \\ &= a \to b \to a^b \to 2 \\ &= f(a^b) = F(2) \qquad (F(1) = F_{a,b}(1) = a^b), \\ a \to b \to 3 \to 3 &= a \to b \to (a \to b \to 2 \to 3) \to 2 \\ &= a \to b \to f(a^b) \to 2 \\ &= f(f(a^b)) = f^2(a^b) = F(3), \\ a \to b \to 4 \to 3 &= a \to b \to (a \to b \to 3 \to 3) \to 2 \\ &= a \to b \to f^2(a^b) \to 2 \\ &= f(f^2(a^b)) = f^3(a^b) = F(4), \\ a \to b \to (c+1) \to 3 &= a \to b \to (a \to b \to c \to 3) \to 2 \\ &= a \to b \to f^{c-1}(a^b) \to 2 \\ &= f(f^{c-1}(a^b)) = f^c(a^b) = F(c+1). \end{aligned}$$

Changing the 4th no. from 3 to 4, we find:

$$\begin{aligned} a \to b \to 2 \to 4 &= a \to b \to (a \to b \to 1 \to 4) \to 3 \\ &= a \to b \to a^b \to 3 \\ &= f^{a\wedge b-1}(a^b) = F(a^b) = g(2) \qquad (g(1) = g_{a,b}(1) = a^b), \\ a \to b \to 3 \to 4 &= a \to b \to (a \to b \to 2 \to 4) \to 3 \\ &= a \to b \to g(2) \to 3 \\ &= f^{g(2)-1}(a^b) = F^2(a^b) = g(3), \\ a \to b \to 4 \to 4 &= a \to b \to (a \to b \to 3 \to 4) \to 3 \\ &= a \to b \to g(3) \to 3 \\ &= f^{g(3)-1}(a^b) = F^3(a^b) = g(4), \\ a \to b \to (c+1) \to 4 &= a \to b \to (a \to b \to c \to 4) \to 3 \\ &= a \to b \to g(c) \to 3 \\ &= f^{g(c)-1}(a^b) = F^c(a^b) = g(c+1). \end{aligned}$$

Changing the 4th no. from 4 to 5, we find:

$$\begin{aligned} a \to b \to 2 \to 5 &= a \to b \to (a \to b \to 1 \to 5) \to 4 \\ &= a \to b \to a^b \to 4 \\ &= g(a^b) = G(2) \qquad\qquad (G(1) = G_{a,b}(1) = a^b), \\ a \to b \to 3 \to 5 &= a \to b \to (a \to b \to 2 \to 5) \to 4 \\ &= a \to b \to g(a^b) \to 4 \\ &= g^2(a^b) = G(3), \\ a \to b \to (c+1) \to 5 &= a \to b \to (a \to b \to c \to 5) \to 4 \\ &= a \to b \to g^{c-1}(a^b) \to 4 \\ &= g^c(a^b) = G(c+1). \end{aligned}$$

Changing the 4th no. from 5 to 6, we find:

$$\begin{aligned} a \to b \to 2 \to 6 &= a \to b \to (a \to b \to 1 \to 6) \to 5 \\ &= a \to b \to a^b \to 5 \\ &= g^{a\wedge b-1}(a^b) = G(a^b) = h(2) \qquad\qquad (h(1) = h_{a,b}(1) = a^b), \\ a \to b \to 3 \to 6 &= a \to b \to (a \to b \to 2 \to 6) \to 5 \\ &= a \to b \to h(2) \to 5 \\ &= g^{h(2)-1}(a^b) = G^2(a^b) = h(3), \\ a \to b \to (c+1) \to 6 &= a \to b \to (a \to b \to c \to 6) \to 5 \\ &= a \to b \to h(c) \to 5 \\ &= g^{h(c)-1}(a^b) = G^c(a^b) = h(c+1). \end{aligned}$$

So, these numbers grow so rapidly that we absolutely need a new recursive function for every 2nd increment of the 4th no. It is best if we need a new one for every single increment of the 4th no. as that would be easier for us.

We therefore take,

$$\begin{aligned} f_{a,b,1}(c) &= a \to b \to c = a \wedge^{(c)} b, \\ f_{a,b,2}(c) &= f_{a,b}(c) = a \to b \to c \to 2 = a \wedge^{(f_{a,b}(c-1))} b = f_{a,b,1}{}^c(1), \\ f_{a,b,3}(c) &= F_{a,b}(c) = a \to b \to c \to 3 = f_{a,b}{}^{c-1}(a^b) = f_{a,b}{}^c(1) = f_{a,b,2}{}^c(1), \\ f_{a,b,4}(c) &= g_{a,b}(c) = a \to b \to c \to 4 = F_{a,b}{}^{c-1}(a^b) = F_{a,b}{}^c(1) = f_{a,b,3}{}^c(1), \\ f_{a,b,5}(c) &= G_{a,b}(c) = a \to b \to c \to 5 = g_{a,b}{}^{c-1}(a^b) = g_{a,b}{}^c(1) = f_{a,b,4}{}^c(1), \\ f_{a,b,6}(c) &= h_{a,b}(c) = a \to b \to c \to 6 = G_{a,b}{}^{c-1}(a^b) = G_{a,b}{}^c(1) = f_{a,b,5}{}^c(1), \end{aligned}$$

and generalise,

$$f_{a,b,(d+1)}(c) = f_{a,b,d}{}^c(1) = a \to b \to c \to (d+1).$$

So, we would need d recursive functions (the absolute minimum is d/2).

What happens when we lengthen the chain from 4 to 5 numbers? First, we try to work out the number

$$\begin{aligned} a \to b \to 2 \to 2 \to 2 &= a \to b \to 2 \to (a \to b \to 2 \to 1 \to 2) \\ &= a \to b \to 2 \to (a \to b \to 2) \\ &= a \to b \to 2 \to (a\hat{}\hat{}b) \\ &= a \to b \to (a \to b \to 1 \to (a\hat{}\hat{}b)) \to (a\hat{}\hat{}b-1) \\ &= a \to b \to a^b \to (a\hat{}\hat{}b-1) \\ &= a \to b \to (a \to b \to (a^b-1) \to (a\hat{}\hat{}b-1)) \to (a\hat{}\hat{}b-2), \end{aligned}$$

which is far larger than anything we encountered before. The 4th no. cannot be reduced below $a\hat{}\hat{}b-1$ since the 3rd no. would seem almost as large as the entire no. itself! We would need $a\hat{}\hat{}b$ recursive functions to work out the answer. Even in the case $a = b = 3$, the 4th no. cannot be reduced below $3\hat{}\hat{}3-1 = 3^{27}-1$ and we would need 3^{27} recursive functions to work out the answer.

$$\begin{aligned} a \to b \to c \to 2 \to 2 &= a \to b \to c \to (a \to b \to c \to 1 \to 2) \\ &= a \to b \to c \to (a \to b \to c) \\ &= a \to b \to c \to (a \,\hat{}^{(c)}\, b), \end{aligned}$$

$a \,\hat{}^{(c)}\, b$ recursive functions would be required to work out the answer.

$$\begin{aligned} a \to b \to c \to 3 \to 2 &= a \to b \to c \to (a \to b \to c \to 2 \to 2) \\ &= a \to b \to c \to (a \to b \to c \to (a \,\hat{}^{(c)}\, b)), \end{aligned}$$

$a \to b \to c \to (a \,\hat{}^{(c)}\, b)$ recursive functions would be required to work out the answer. The increment of the 4th digit from 2 to 3 would mean that, when reduced to a chain of 4 numbers, the 4th no. alone would become so large that $a \,\hat{}^{(c)}\, b$ recursive functions would be required to work out just that number!

At this stage the numbers become too large to even reduce the length of the chain by one number without terribly complicated mathematics, too complex to describe here. So, at this level, we think it is best to explore the chained arrow notation and just try to imagine how much bigger these numbers are and how much closer to the infinite.

Moving up to a chain of 6 numbers, if all but the first 2 nos. contain 2's, then

$$\begin{aligned} & a \to b \to 2 \to 2 \to 2 \to 2 \\ &= a \to b \to 2 \to 2 \to (a \to b \to 2 \to 2 \to 1 \to 2) \\ &= a \to b \to 2 \to 2 \to (a \to b \to 2 \to 2) \\ &= a \to b \to 2 \to 2 \to (a \,\hat{}^{(a\hat{}b)}\, b) \\ &= a \to b \to 2 \to (a \to b \to 2 \to 1 \to (a \,\hat{}^{(a\hat{}b)}\, b)) \to (a \,\hat{}^{(a\hat{}b)}\, b - 1) \\ &= a \to b \to 2 \to (a \to b \to 2) \to (a \,\hat{}^{(a\hat{}b)}\, b - 1) \\ &= a \to b \to 2 \to (a\hat{}\hat{}b) \to (a \,\hat{}^{(a\hat{}b)}\, b - 1) \end{aligned}$$

$= a \to b \to 2 \to (a \to b \to 2 \to (a\hat{}\hat{}b-1) \to (a\ \hat{}^{(a\hat{}b)}\ b - 1)) \to (a\ \hat{}^{(a\hat{}b)}\ b - 2),$

which means that the 5th no. cannot be reduced beyond $a\ \hat{}^{(a\hat{}b)}\ b$ (let's ignore the -1 from now on) and in the case $a = b = 3$, the 5th no. cannot be reduced beyond $3\ \hat{}^{(27)}\ 3$.

Similarly, with a chain of 7 numbers,

$a \to b \to 2 \to 2 \to 2 \to 2 \to 2$

$= a \to b \to 2 \to 2 \to 2 \to (a \to b \to 2 \to 2 \to 2 \to 1 \to 2)$

$= a \to b \to 2 \to 2 \to 2 \to (a \to b \to 2 \to 2 \to 2)$

$= a \to b \to 2 \to 2 \to 2 \to (a \to b \to 2 \to (a\hat{}\hat{}b)),$

we can now see a pattern. The 6th no. cannot be reduced below a no. equal to a similar chain with 5 nos. (just 2 less than 7) or $a \to b \to 2 \to (a\hat{}\hat{}b)$, which shows how much greater the gulf is between successive lengthenings of the chain, even by just one number!

In general, with a chain of n numbers, where all but the first 2 nos. contain 2's,

$a \to b \to 2 \to 2 \to \ldots\ldots \to 2 \to 2 \to 2$

$= a \to b \to 2 \to 2 \to \ldots\ldots \to 2 \to (a \to b \to 2 \to 2 \to \ldots\ldots \to 2 \to 1 \to 2)$

$= a \to b \to 2 \to 2 \to \ldots\ldots \to 2 \to (a \to b \to 2 \to 2 \to \ldots\ldots \to 2),$

which means that the (n-1)th no. cannot be reduced below a no. equal to a similar chain with n-2 numbers.

Of course, there is no limit as to how long the chain can be; there are also numbers that are so huge that they cannot even be expressed using Conway's chained arrow notation because the chains are too long. In order to express these numbers, yet another new notation needs to be used.

Bird's Multiple Right-arrow Notation

I intend myself to go another step further. As with Knuth's up-arrow notation, which dealt with power towers that are too high, I intend to do the same with chains that are too long, by inventing a system of multiple right-arrows. It works similarly to Knuth's as follows:

$a \to\to b \quad = \quad a \to a \to a \to \ldots\ldots \to a$ (b terms),

$a \to\to\to b \quad = \quad a \to\to a \to\to a \to\to \ldots\ldots \to\to a$ (b terms),

$a \to^{4} b \quad = \quad a \to\to\to\to b$

$= a \to^{3} a \to^{3} a \to^{3} \ldots\ldots \to^{3} a$ (b terms),

and so on. In general, for all $n \geq 2$,

$a \to^{n} b \quad = \quad a \to\to\to \ldots\ldots \to b$ (with n right-arrows)

$= a \to^{n-1} a \to^{n-1} a \to^{n-1} \ldots\ldots \to^{n-1} a$ (b terms), so, these operators are also recursive.

It is like going back to the beginning of this document and starting all over again but rotating the arrows 90 degrees clockwise.

As with multiple up-arrows, when working with multiple right-arrows, we start from the right and move leftwards as this produces a number far, far greater than the one produced by working from left to right (these operations are also neither commutative nor associative). For example, for all $n \geq 2$,

$$a \to^{n+1} 4 \;=\; a \to^{n} a \to^{n} a \to^{n} a \;=\; a \to^{n} (a \to^{n} (a \to^{n} a)).$$

Even with a, b small, as one increases the number of right-arrows, even the numbers representing the lengths of the chains grow with astonishing rapidity. For example:

$$\begin{aligned}
3 \to 3 \quad &= \; 3 \,\hat{}\, 3 \;=\; 27, \\
3 \to\to 3 \quad &= \; 3 \to 3 \to 3 \;=\; 3 \,\hat{}\hat{}\hat{}\, 3 \;=\; 3 \,\hat{}\hat{}\, 3^{27}, \\
3 \to\to\to 3 \quad &= \; 3 \to\to 3 \to\to 3 \\
&= \; 3 \to\to (3 \to 3 \to 3) \\
&= \; 3 \to\to (3\hat{}\hat{}3^{27}) \\
&= \; 3 \to 3 \to 3 \to \ldots\ldots \to 3
\end{aligned}$$

(with $3\hat{}\hat{}3^{27}$ terms, in other words, this chain is so long that it has length $3\hat{}\hat{}3^{27}$),

$$\begin{aligned}
3 \to^{4} 3 \quad &= \; 3 \to^{3} 3 \to^{3} 3 \\
&= \; 3 \to^{3} (3 \to\to 3 \to\to 3) \\
&= \; 3 \to^{3} (3 \to\to (3\hat{}\hat{}3^{27})) \\
&= \; 3 \to^{3} (3 \to 3 \to 3 \to \ldots\ldots \to 3)
\end{aligned}$$

(the chain inside the brackets has length $3\hat{}\hat{}3^{27}$)

$$= \; 3 \to\to 3 \to\to 3 \to\to 3 \to\to \ldots\ldots \to\to 3$$

(so huge that even this double right-arrowed chain has length $3 \to 3 \to 3 \to \ldots\ldots \to 3$, which itself is a chain of length $3\hat{}\hat{}3^{27}$).

If it's this big after just 4 right-arrows, consider how rapidly the numbers grow if they go up in this sequence:

$f(0) = 1$, $f(1) = 3 \to 3 = 27$, $f(2) = 3 \to^{27} 3$, $f(3) = 3 \to^{f(2)} 3$, and so on,

with $f(n+1) = 3 \to^{f(n)} 3$.

It would get so huge, we would have to invent yet another new notation.

Bird's Chained Down-arrow Notation

This works in exactly the same way as Conway's chained (right) arrow notation with the only difference being that all the arrows have been rotated by 90 degrees clockwise. The important note is that when the chain consists of 3 numbers, then

$a \downarrow b \downarrow c = a \rightarrow^{c} b$ (with c right-arrows).

This means that,

$$\begin{aligned}
3 \downarrow 3 \downarrow 1 \downarrow 2 &= 3 \downarrow 3 = 3 \rightarrow 3 = 27 = f(1),\\
3 \downarrow 3 \downarrow 2 \downarrow 2 &= 3 \downarrow 3 \downarrow (3 \downarrow 3 \downarrow 1 \downarrow 2)\\
&= 3 \downarrow 3 \downarrow 27\\
&= 3 \rightarrow^{27} 3 = f(2),\\
3 \downarrow 3 \downarrow 3 \downarrow 2 &= 3 \downarrow 3 \downarrow (3 \downarrow 3 \downarrow 2 \downarrow 2)\\
&= 3 \downarrow 3 \downarrow (3 \rightarrow^{27} 3)\\
&= 3 \rightarrow^{f(2)} 3 = f(3),\\
3 \downarrow 3 \downarrow (n+1) \downarrow 2 &= 3 \downarrow 3 \downarrow (3 \downarrow 3 \downarrow n \downarrow 2)\\
&= 3 \downarrow 3 \downarrow (3 \rightarrow^{f(n-1)} 3)\\
&= 3 \rightarrow^{f(n)} 3 = f(n+1),
\end{aligned}$$

and $3 \downarrow 3 \downarrow n \downarrow 2$ grows as rapidly as f(n) as defined on the previous page.

But there are also numbers that are so huge that they cannot even be expressed using my chained down-arrow notation because even those chains are too long. Oh well, back to the beginning again! I would need to rotate the arrows another 90 degrees clockwise.

Further new notations

The next step for me would be to invent Bird's Multiple Down-arrow Notation similar to Knuth's in which for all $n \geq 2$,

$$\begin{aligned}
a \downarrow^{n} b &= a \downarrow\downarrow\downarrow \ldots\ldots \downarrow b && \text{(with n down-arrows)}\\
&= a \downarrow^{n-1} a \downarrow^{n-1} a \downarrow^{n-1} \ldots\ldots \downarrow^{n-1} a && \text{(b terms).}
\end{aligned}$$

We find that,

$$\begin{aligned}
3 \downarrow 3 &= 3 \rightarrow 3 = 27,\\
3 \downarrow\downarrow 3 &= 3 \downarrow 3 \downarrow 3\\
&= 3 \rightarrow\rightarrow\rightarrow 3\\
&= 3 \rightarrow 3 \rightarrow 3 \rightarrow \ldots\ldots \rightarrow 3 && \text{(with length 3\^{}\^{}3}^{27}\text{),}
\end{aligned}$$

$$\begin{aligned} 3 \downarrow\downarrow\downarrow 3 &= 3 \downarrow\downarrow 3 \downarrow\downarrow 3 \\ &= 3 \downarrow\downarrow (3 \rightarrow\rightarrow\rightarrow 3) \\ &= 3 \downarrow 3 \downarrow 3 \downarrow \ldots\ldots \downarrow 3 \end{aligned}$$

(with length $3 \rightarrow 3 \rightarrow 3 \rightarrow \ldots\ldots \rightarrow 3$, itself of length $3{\wedge}{\wedge}3^{27}$).

Then, for even greater numbers, I would invent Bird's Chained Left-arrow Notation, which again works in exactly the same way as my chained down-arrow notation except that all the arrows have been rotated clockwise by another 90 degrees, with

$$a \leftarrow b \leftarrow c = a \downarrow^{c} b \qquad \text{(with c down-arrows),}$$

and again finding that,

$$\begin{aligned} 3 \leftarrow 3 \leftarrow (n+1) \leftarrow 2 &= 3 \leftarrow 3 \leftarrow (3 \leftarrow 3 \leftarrow n \leftarrow 2) \\ &= 3 \leftarrow 3 \leftarrow (3 \downarrow^{g(n-1)} 3) \\ &= 3 \downarrow^{g(n)} 3 = g(n+1), \end{aligned}$$

where $g(1) = 3 \leftarrow 3 = 3 \downarrow 3 = 3 \rightarrow 3 = 27$.

When even these chains get too large, I yet again rotate the arrows another 90 degrees clockwise and go back to the beginning again. Of course, each movement of the arrows brings us into a whole new world of numbers. In fact, I can continue to rotate the arrows and introduce further new notation. The next step is Bird's Multiple Left-arrow Notation and then back to up-arrows again, and these arrows would have gone round one full revolution. In order to cope with going back to the same symbols again, I need to combine all of the different notations introduced thus far and extend it. This is achieved in the following section.

Bird's Revolving Arrow Notation

This takes me yet another giant leap further. We have seen that for single arrowed chains of length 3,

$$\begin{aligned} a \rightarrow b \rightarrow c &= a \wedge^{(c)} b, \\ a \downarrow b \downarrow c &= a \rightarrow^{c} b, \\ a \leftarrow b \leftarrow c &= a \downarrow^{c} b. \end{aligned}$$

When the arrows return to the up position again, I need to distinguish them from the original up-arrows. I do this by inserting a subscript between the arrow and the superscript (as the word processor does not allow the subscript and the superscript to be neatly on top of each other). While the superscript (c) denotes the number of like arrows, the new subscript denotes the number of revolutions that the arrow has made. The next equation in this sequence is therefore,

$$a \uparrow_1 b \uparrow_1 c = a \leftarrow^{c} b,$$

and the two single up-arrows on the left hand side have each completed 1 revolution.

So, the revolving arrows are defined as follows. This is the multiple arrow notation similar to Knuth's:

$$a \uparrow_r^{n} b \quad = \quad a \uparrow_r^{n-1} a \uparrow_r^{n-1} a \uparrow_r^{n-1} \ldots\ldots \uparrow_r^{n-1} a \qquad \text{(b terms),}$$

$$a \rightarrow_r^{n} b \quad = \quad a \rightarrow_r^{n-1} a \rightarrow_r^{n-1} a \rightarrow_r^{n-1} \ldots\ldots \rightarrow_r^{n-1} a \qquad \text{(b terms),}$$

$$a \downarrow_r^{n} b \quad = \quad a \downarrow_r^{n-1} a \downarrow_r^{n-1} a \downarrow_r^{n-1} \ldots\ldots \downarrow_r^{n-1} a \qquad \text{(b terms),}$$

$$a \leftarrow_r^{n} b \quad = \quad a \leftarrow_r^{n-1} a \leftarrow_r^{n-1} a \leftarrow_r^{n-1} \ldots\ldots \leftarrow_r^{n-1} a \qquad \text{(b terms),}$$

where $n \geq 2$ and r is the no. of revolutions that the arrows have made (if $r = 0$, it can be omitted and the operation $a \uparrow^{n} b$ is exactly the same as $a \,\hat{}^{(n)}\, b$).

This is the chained arrow notation (similar to Conway's which works to the same rules) when the arrows rotate by 90 degrees:

$$a \rightarrow_r b \rightarrow_r c \quad = \quad a \uparrow_r^{c} b,$$

$$a \downarrow_r b \downarrow_r c \quad = \quad a \rightarrow_r^{c} b,$$

$$a \leftarrow_r b \leftarrow_r c \quad = \quad a \downarrow_r^{c} b,$$

$$a \uparrow_{r+1} b \uparrow_{r+1} c \quad = \quad a \leftarrow_r^{c} b,$$

where c is the no. of like arrows on the right hand side and r is the no. of revolutions that the arrows have made (except the two single up-arrows on the left hand side, where it is r+1).

So, my Bird's Revolving Arrow Notation (BRAN), like Knuth's up-arrow and Conway's chained arrow notations on which it is based, once again involves the use of additional recursion. It is now possible to get closer to infinity than anyone else has ever done by increasing the number of revolutions of the arrows. Let's make some of my own hyper-numbers, numbers only expressible with BRAN, and see how far we can go.

Let's take $\quad N = 3 \uparrow_G^{4} 3,$

where the 4 up-arrows have made so many revolutions that they have each made a Graham's Number of them. If that is not enough, how about N up-arrows, each having made N revolutions, with two more N's on either side of the arrows. And that's just the start of it! For,

$$X(1) = N \uparrow_N^{N} N,$$

$$X(2) = X(1) \uparrow_{X(1)}^{X(1)} X(1),$$

$$X(3) = X(2) \uparrow_{X(2)}^{X(2)} X(2), \qquad \text{and so on, especially noting}$$

$$X(N) = X(N\text{-}1) \uparrow_{X(N-1)}^{X(N-1)} X(N\text{-}1).$$

And far beyond that, because with this X function renamed X_1 and taking $X_2(1) = X_1(N)$, we next find

$$X_2(2) = X_1^{2}(N) = X_1(X_1(N)), \qquad X_2(3) = X_1^{3}(N), \qquad X_2(4) = X_1^{4}(N),$$

and so on, taking $\quad X_3(1) = X_2(N) = X_1^{N}(N).$

Then, we next find,

$X_3(2) = X_2^2(N)$, $X_3(3) = X_2^3(N)$, $X_3(4) = X_2^4(N)$, and so on, taking
$X_4(1) = X_3(N) = X_2^N(N)$.

It follows that,

$X_3(n) = X_2^n(N) = X_2(X_2^{n-1}(N)) = X_2(X_3(n-1)) = X_1^{X_3(n-1)}(N)$,
$X_3(n+1) = X_1^{X_3(n)}(N)$,

which shows just how rapidly they grow.

Then we continue this sequence,

$X_4(2) = X_3^2(N)$, $X_4(3) = X_3^3(N)$, $X_4(4) = X_3^4(N)$, and so on, taking
$X_5(1) = X_4(N) = X_3^N(N)$.

As we keep on adding new recursive functions,

$X_5(n) = X_4^n(N)$, $X_6(n) = X_5^n(N)$, $X_7(n) = X_6^n(N)$, and so on,

making another generalisation,

$X_{r+1}(n) = X_r^n(N)$, $X_{r+2}(n+1) = X_r^{X_{r+2}(n)}(N)$,

we can produce the hyper-numbers $H = X_N(N)$ and $X_H(N)$.

Then the subscript (H) in $X_H(N)$ gets replaced by a copy of $X_H(N)$, where the H itself gets continually replaced by copies of $X_H(N)$, and we repeat this process $X_H(N)$ times, denoting the final hyper-number by Bird's Number (B). Is this a new world record for expressing the largest finite number?

How much further do we need to go before we reach infinity (which is produced just by dividing 1 by 0)? Are we still scratching the surface?

References

Knuth's Up-arrow Notation, Graham's Number and Conway's Chained Arrow Notation were used from Robert Munafo's web-page 'Large Numbers', which can be found at this address:

http://www.mrob.com/pub/largenum.html

Author of Report: Christopher Michael Bird.
Date started: 13th December 2000.
Date completed: 20th December 2000.